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NONLINEAR WAVES ON THE SURFACE OF A LIQUID FILM RUNNING DOWN A VERTICAL WALL

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It is known from experiments [1, 2] that the flow of a liquid film down a vertical plane has a wave character even for small Reynolds numbers. This is related to the fact that the flow of a film of thickness h with a plane free surface whose velocity profile is semiparabolic, $u = 3u_0(y/h - y^2/2h^2)$, is unstable starting from very small Reynolds numbers, i.e., infinitesimal long-wave disturbances increase exponentially with time [3, 4]. As a result of nonlinear effects stationary periodic and soliton flow regimes may be formed. Since a complete treatment of such problems is extremely complicated, various simplifications are used to solve it.

Thus, for low flow rates (Re $\sim$ 1) the problem of wave regimes can be reduced to that of solving a single equation for the film thickness [5]. However, stationary traveling waves are practically not observed at these flow rates, and although the form of the solutions of this equation [6] is in good qualitative agreement with the form of the waves observed in experiment, there is no quantitative agreement. A similar situation occurs also with a two-wave equation [7] which contains only quadratic nonlinear term, and therefore describes the

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behavior of only weakly nonlinear waves for moderate flow rates. For such flow rates (Re ∿ 10-100) experiments show that there are nonlinear waves [2] whose amplitude is of the same order of magnitude as the average film thickness, and which cannot be described by taking account of only a quadratic nonlinearity.

Under the assumption that the longitudinal velocity profile is self-similar

$$u = 3U(x, t)(y/h(x, t) - y^2/2h^2(x, t)) \tag{1}$$

and that the waves are long, a system of equations was derived [7, 8] to describe the behavior of disturbances on a film for moderate values of Re:

$$\frac{\partial q}{\partial t} + 1{,}2\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) = -\frac{3\nu}{h^2}q + gh + \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3}, \quad \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{2}$$

where q is the instantaneous flow rate of the liquid in the cross section x, h is the instantaneous film thickness, g is the acceleration due to gravity, and σ is the surface tension.

Weakly nonlinear periodic solutions of system (2) were found in [8, 9], and negative solitons for which $\int\limits_{-\infty}^{\infty}(h - h_\infty)\,dx < 0$, where h_∞ is the thickness of the undisturbed film, were found in [10].

The purpose of the present article is to find strongly nonlinear periodic stationary waves which in the limit as the wave number α → 0 pass over into positive soliton solutions $\left(\int\limits_{-\infty}^{\infty}(h - h_\infty)\,dx > 0\right)$, and to compare them with experiment.

For a stationary traveling wave

$$h = h(\xi),\ q = q(\xi),\ \xi = x - ct \tag{3}$$

where c is the phase velocity of the wave, we obtain from the second of Eqs. (2)

$$q = q_0[1 + (ch_0/q_0)(h/h_0 - 1)], \tag{4}$$

where q_0 is the flow rate in the cross section where $h = h_0$.

Choosing these quantities as characteristic, and using them to make Eqs. (2)-(4) dimensionless, we obtain (omitting the dimensionless sign)

$$\begin{gathered} -c^2h' + 2.4c(1 + c(h-1))h'/h - 1.2(1 + c(h-1))^2h'/h^2 \\ = -3(1 + c(h-1))/\mathrm{Re}\,h^2 + h/\mathrm{Fr} + \mathrm{We}\,hh''', \\ \mathrm{Re} = q_0/\nu, \quad \mathrm{Fr} = q_0^2/gh_0^3, \quad \mathrm{We} = \sigma h_0/\rho q_0^2. \end{gathered} \tag{5}$$

Primes denote differentiation with respect to ξ.

For periodic solutions we take as h_0 the thickness averaged over a wavelength λ:

$$h_0 \equiv \langle h \rangle = \frac{1}{\lambda}\int\limits_0^\lambda h\,d\xi.$$

It is clear from (4) that in this case $q_0 \equiv \langle q \rangle$. For soliton solutions we take as h_0 the value of h at infinity.

In a wave-free film h = 1, and

$$q_0 = gh_0^3/3\nu \to \mathrm{Fr} = \mathrm{Re}/3.$$

By neglecting nonlinear terms, the dimensionless Eqs. (2) yield the result of the linear theory: disturbances of the form $\exp[i\alpha(x - ct)]$ are unstable for $\alpha < \alpha_H = \sqrt{3/\mathrm{We}}$. If we introduce the new coordinate $\xi_1 = \xi\sqrt{3/\mathrm{We}}$, for H = h − 1 Eq. (5) takes the form

$$\begin{gathered} (cz - 3F)H + (0{,}2c^2 - 1{,}2(c-1)^2)H' - 3H''' = F - z + \\ + 3F(H^2 + H^3/3) - 0{,}4c^2H'(H + H^2/2) + 9H'''(H + H^2 + H^3/3), \end{gathered} \tag{6}$$

where

$$z = \sqrt{3\mathrm{We}/\mathrm{Re}^2};\ F = \sqrt{\mathrm{We}/3\mathrm{Fr}^2}.$$

Making this change normalizes the range of unstable wave numbers to unity. Using the fact that $\langle H \rangle = 0$, we find from (6) a relation between the parameters F and z for periodic solutions:

$$F = \frac{z - 9\langle H'''(H + H^2 + H^3/3)\rangle}{1 + 3\langle H^2 + H^3/3\rangle}.$$

Since soliton solutions are the limiting forms for periodic solutions as $\lambda \to \infty$, for them $F = z$.

Thus the problem is reduced to that of finding periodic and soliton solutions of Eq. (6). The phase velocity c is the eigenvalue, and z is a parameter. We convert to experimentally measurable quantities by the formulas

$$\mathrm{Re} = (81\sigma^3 F/g\rho^3\nu^4 z^7)^{1/11},\quad c^* = c(\sigma^2 g^3 \nu/3\rho^2 F^3 z)^{1/11},$$
$$\lambda^* = \frac{2\pi}{\alpha}(\sigma^4\nu^2 F^5/9\rho^4 g^5 z^2)^{1/11},\quad A^* = A(243\sigma\nu^6 F^4/\rho g^4 z^6)^{1/11},$$

where c^*, λ^*, and A^* are the dimensional phase velocity wavelength and amplitude of the wave.

We seek a periodic wave with a wave number α in the form

$$H = \sum_{-\infty}^{\infty} H_n \exp[i\alpha n \xi_1]. \tag{7}$$

Since H is a real function, $H_n = \bar{H}_{-n}$, where the bar denotes the complex conjugate.

Retaining the first N/2 harmonics in (7), and substituting them into Eq. (6), we obtain a system of N + 1 equations for N + 3 unknowns ($F, c, H_0, H_{\pm 1}, \ldots, H_{\pm N/2}$):

$$H_n = \frac{3F\varphi_n - 0.4c^2\psi_n + 9\chi_n}{cz - 3F + i\alpha n(0.2c^2 - 1{,}2(c-1)^2) + 3i\alpha^3 n^3}, \quad n = 0, \pm 1, \ldots, \pm N/2. \tag{8}$$

Here φ_n, ψ_n, and χ_n are, respectively, the Fourier harmonics of the functions

$$\varphi = H^2 + H^3/3,\ \psi = H'(H + H^2/2),\ \chi = H'''(H + H^2 + H^3/3).$$

In view of the normalization of the function H, $H_0 \equiv \langle H \rangle = 0$. In addition, we can always choose the point $\xi_1 = 0$ so that, for example, $\mathrm{Im}(H_1) = 0$.

Thus, system (8) consists of N + 1 nonlinear equations in N + 1 unknowns. For a specified value of α the calculation was begun for large values of z, using the results from [11, 12] as a first approximation. Smaller values of z were reached by requiring continuity. For each value of α there is a critical value z_* such that for $z < z_*$ the solution cannot be found, at least by this method. The fast Fourier transform procedure (13) was used to calculate the Fourier transforms of the functions and their inverses [13]. In breaking off series (7), the number of harmonics was taken so as to satisfy the relation

$$|H_{N/2}| \Big/ \sup_{n \leqslant N/2} |H_n| < 10^{-3}.$$

Depending on the values of α and z, the number N was varied from 16 to 128.

Figure 1 shows the z dependence of the phase velocity and amplitude of the waves for several values of the wave numbers; curves 1-3 correspond to $\alpha = 0.2$, 0.35, and 0.5, and curves 4 to the soliton solutions. The points represent data on soliton solutions of the two-wave equation with a quadratic nonlinearity [7] obtained in [11, 12]. It can be seen from the graph that for large values of z the waves are weakly nonlinear, and that the results are in good agreement.

Figures 2 and 3 compare the calculated (lines) profiles of strongly nonlinear waves for a water-glycerine film ($\nu = 4.9 \cdot 10^{-6}$ m^2/sec, $\sigma/\rho = 59 \cdot 10^{-6}$ m^3/sec^2, Re = 7.2) with the experimental (points) values from [14]. In Fig. 2, $\lambda = 34.3$ mm, $c_{exp} = 320$ mm/sec, $c_{calc} = 318$ mm/sec; in Fig. 3, $\lambda = 18.5$ mm, $c_{calc} = 262$ mm/sec, $c_{exp} = 270$ mm/sec. Clearly there is good quantitative agreement.

A similar comparison for water ($\nu = 1.03 \cdot 10^{-6}$ m^2/sec, $\sigma/\rho = 72.9 \cdot 10^{-6}$ m^3/sec^2, Re = 9.8) is given in Fig. 4, where $\lambda = 36.8$ mm, $c_{calc} = 260$ mm/sec, and $c_{exp} = 232$ mm/sec (experimental values from [2]). The agreement is somewhat worse, but quite satisfactory.

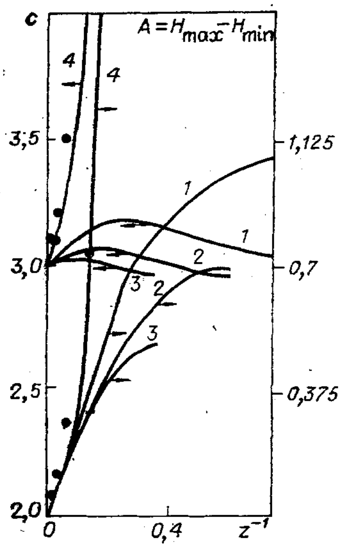

Fig. 1

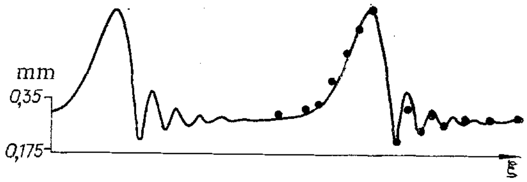

Fig. 2

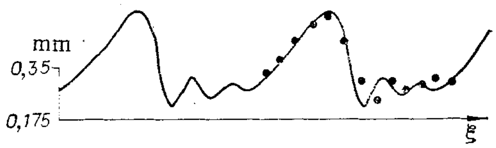

Fig. 3

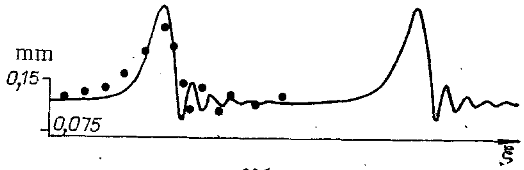

Fig. 4

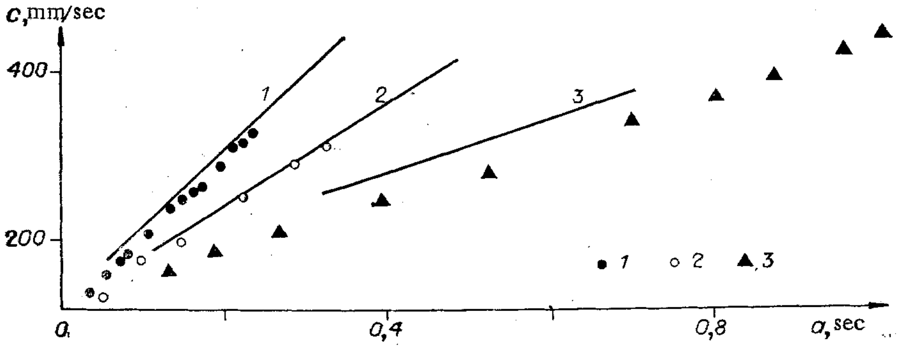

Fig. 5

It can be seen from Figs. 2-4 that the greatest differences between the calculated and experimental values occur on the oscillating leading edge of the waves. Apparently this is related to the fact that assumption (1) is rather crude for a detailed description of the fine structure.

Figure 5 summarizes the experimental data [2, 15] on the dependence of the wave velocity on amplitude for water ($\nu = 1.03 \cdot 10^{-6}$ m/sec, $\sigma/\rho = 72.9 \cdot 10^{-6}$ m^3/sec — curve 1) and water–glycerine solutions (curve 2 — $\nu = 2.06 \cdot 10^{-6}$ m^2/sec, $\sigma/\rho = 40.3 \cdot 10^{-6}$ m^3/sec^2; curve 3 — $\nu = 11.2 \cdot 10^{-6}$ m^2/sec, $\sigma/\rho = 55.9 \cdot 10^{-6}$ m^3/sec^2). The corresponding points 1-3 represent the calculated values. Although the values of z and α for a given material were varied in the calculations, it is clear that the calculated points lie practically on straight lines.

In the calculations solutions were found with negative values of the velocity of the liquid in certain cross sections, corresponding to retrogressive flow. In other solutions regimes were found in which the velocity of the liquid at the wave crests is higher than the phase velocity, corresponding to supercritical flow. It is clear from (1) and (4) that these situations arise, respectively, for cross sections in which the deviations from the average level satisfy the inequalities $h < h_{min} = -1/c$, $h > h_{max} = (2c - 3)/c$.

It should be noted that in the transition from normal wave regimes to regimes with retrogressive flows in the troughs, and to supercritical regimes, there are no changes of the dependence of the wave amplitudes on the velocity. Thus, in Fig. 5 for water these regimes correspond to calculated points with an amplitude $a \gtrsim 1$.

So far such regimes have not been detected experimentally. This may be due to the fact that the supercritical and retrogressive zones occupy narrow portions on the wavelength, and are difficult to fix, and possibly the fact that they appear in the calculations as a result of making the simplifying assumption (1).

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